# Constructions in Symplectic and Contact Geometry

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*These are notes compiled from the Constructions in Symplectic and Contact Geometry Student Seminar at UC Berkeley, Spring 2025. The seminar's goal is to build a shared knowledge base of ways to cook up interesting properties in contact and symplectic manifolds. As is the nature of a seminar, these accompanying notes are very rough and may contain typos, approximations, and errors. If any are found, feel free to email me with any corrections.*

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## Contents

# 1 Convex Surfaces

## 1.1 First Definition of a Convex Surface

To motivate our discussion, we consider two fundamental questions in contact geometry:

**Question 1.** Which 3-manifolds admit contact structures?

**Question 2.** Given a fixed contact 3-manifold $(M, \xi)$, how many distinct contact structures does $M$ support?

A key tool in addressing these questions is convex surface theory, which provides a way to analyze and classify contact structures using surfaces embedded in 3-manifolds.

**Theorem 1.1** (Lickorish)**.** *Every "nice" 3-manifold can be obtained via surgery along a knot or link in $S^3$. The key idea behind this construction is the process of cutting and gluing pieces together to form new manifolds.*

**Example.** Standard examples of contact 3-manifolds include:

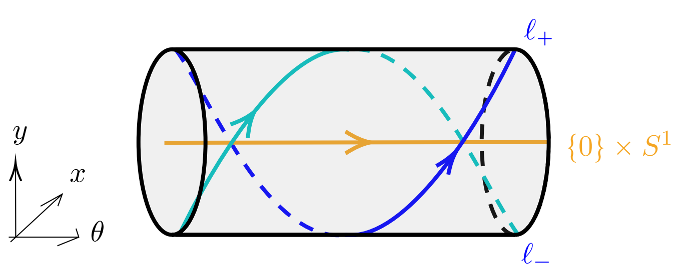
- The solid torus $D^2 \times S^1$, which can be described in coordinates as:

$$D^2 \times S^1 = \{(x, y, \theta) \in \mathbb{R}^2 \times S^1 \mid x^2 + y^2 \leqslant 1\}$$

equipped with the contact form $\alpha = \cos(\theta)dx - \sin(\theta)dy$. From $\alpha$, we have two distinguished dividing curves on the boundary of the neighborhood, given by

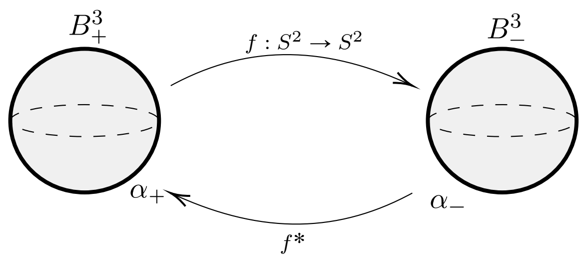
$$\ell_\pm := \{(\pm\delta \sin(z), \pm\delta \cos(z), z) \mid \theta \in S^1\}.$$

These curves demonstrate the twisting of the contact plane as you move around the solid torus in the direction of $\theta$.



- The 3-sphere $S^3$, which can be decomposed into two solid 3-balls glued together along their boundary:

$$S^3 = B^3_+ \cup_f B^3_-.$$




The process of gluing 3-manifolds together can be technically challenging. Convex surface theory provides a way to manage this difficulty by working with surfaces that interact with the contact structure in a relatively manageable way.
We'll begin our discussion of convex surface theory with the notion of characteristic foliations.

**Definition 1.2** (Characteristic foliation)**.** Given a contact 3-manifold $(M^3, \xi)$, an embedded surface $\Sigma \subset M$ has a **characteristic foliation** $\mathcal{F}_\xi$ defined by the intersection of the contact structure with the tangent bundle of the surface:

$$\mathcal{F}_\xi = T\Sigma \cap \xi.$$

**Remark 1.** At most points on $\Sigma$, the intersection $T_p\Sigma \cap \xi_p$ is one-dimensional, forming a smooth distribution that defines a foliation of $\Sigma$. However, at singular points $p$ where $T_p\Sigma$ aligns with $\xi_p$, the characteristic foliation exhibits isolated singularities.

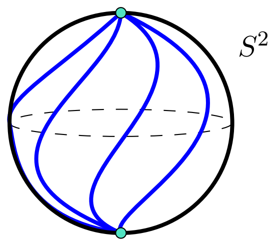
**Remark 2.** Some authors (see [Gei08, Dfn 2.5.18]) alternatively define the characteristic foliation using the symplectic orthogonal complement:

$$\mathcal{F}_\xi = (T\Sigma \cap \xi)^\perp.$$

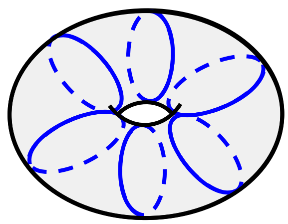
In this formulation, singularities of the foliation correspond to points where the distribution becomes zero-dimensional.

**Example.** Common examples of surfaces with their characteristic foliations include:

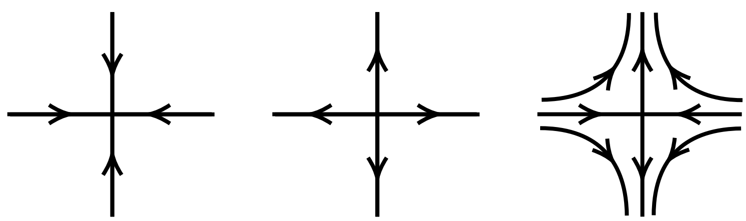
- A 2-sphere $S^2$ embedded in $\mathbb{R}^3$ with the radial contact structure. See [Gei08, Exm 2.5.19].




- A torus $T^2$ embedded in the solid torus $D^2 \times S^1$.



Singularities of the characteristic foliation in a neighborhood can look like:



One might ask, is it possible to have singularities with an arbitrary number of prongs, such as 6 instead of the usual 4? In fact, there is an analogy with Morse functions on surfaces that suggests that singularities of characteristic foliations must follow certain constraints. Generically, A small perturbation of $\Sigma$ can be performed to ensure that the intersection with $\xi$ always produces a foliation with only the above expected types of singularities.
The characteristic foliation is of importance precisely because $\mathcal{F}_\xi$ on $\Sigma$ determines the contact structure in a neighborhood of $\Sigma$. This means that by understanding the foliation, we can infer information about the local structure of $\xi$. Before we proceed further in developing this tool, we will first address a more foundational question:

**Question.** What does it *really* mean to glue two 3-manifolds together?

Recall that in the smooth category, we **glue** two manifolds $M$ and $N$ together along some submanifold $\Sigma$ via an identification of a neighborhood of $\Sigma$ in each $M$ and $N$. More precisely, if given an embedding $\Sigma \hookrightarrow M$ and $\Sigma \hookrightarrow N$, then we form $M \cup_\Sigma N$ by taking the fibered product:
All that's needed to make this identification is a diffeomorphism $\phi : \Sigma \subset M \to \Sigma \subset N$. However, to extend this notion to the contact setting, we'll need $\phi$ to preserve the contact structure along $\Sigma$. That is, we need a way of matching the contact structures on $N$ and $M$ via $\Sigma$. To analyze how contact structures behave under gluing, we must introduce the notion of contact vector fields:

**Definition 1.3** (Contact vector field)**.** A vector field $v \in \Gamma(TM)$ is **contact** if it satisfies:

$$\mathcal{L}_v(\alpha) = f\alpha$$

for some smooth function $f \in C^\infty(M)$. Equivalently, this condition ensures that the flow generated by $v$ preserves the contact structure, meaning $(\phi_V)_*(\xi) = \xi$.

**Definition 1.4** (**Convex surface**)**.** A closed surface $\Sigma \subset M^3$ is **convex** (with respect to a contact vector field $v$) if $v$ is transverse to $\Sigma$.

A convex surface naturally decomposes into regions separated by a dividing set:

**Definition 1.5** (Dividing set)**.** The **dividing set** of a closed surface $\Sigma \subset M^3$ is the locus where the contact form evaluates to zero along $v$:

$$\Gamma = \{p \in \Sigma \mid \alpha_p(v_p) = 0\}.$$

Equivalently, it consists of points where the vector field $v$ lies in the contact plane $\xi_p$.

A key question is how the dividing set relates to the characteristic foliation:

**Question.** What is the relationship between the dividing set and the characteristic foliation of a surface?

**Answer.** Given a closed surface $\Sigma \subset M$, it naturally inherits a characteristic foliation from the contact structure $\xi$. A fundamental result states that any such surface can be slightly perturbed to become convex, meaning it admits a transverse contact vector field. The dividing set of $\Sigma$ is determined by the equation $\iota_v(\alpha) = 0$, i.e. a level set of a function $\Sigma \to \mathbb{R}$. One can show that 0 is a regular value, so that $(\iota_v(\alpha))^{-1}(0)$ is a hypersurface on $\Sigma$ - a 1-dimensional (multi)curve. Moreover, one may show that $\Gamma_\xi$ is nonempty and is transverse to the characteristic foliation of $\Sigma$.

**Lemma 1.6.** $\Gamma_\xi$ is a submanifold of $\Sigma$.

**Proof.** Without loss of generality, we can identify a tubular neighborhood of $\Sigma$ as $\nu(\Sigma) \simeq \Sigma \times I_z$, and in these coordinates we can write $\alpha = \beta + udz$. We also choose the identification so that $\partial_z$ corresponds to the contact vector field $v$. Consider $\Gamma_\xi := \{p \in \Sigma \mid \alpha_p(v_p) = 0\}$. So if $p \in \Gamma_x i$, then

$$0 = \alpha_p(v_p) = \beta_p(\partial_z) + u(p)dz(\partial_z) = 0 + u(p),$$

so that $\Gamma_\xi = u^{-1}(0)$. We can thus show that $\Gamma_x i$ is a submanifold of $\Sigma$ if 0 is a regular value of $u$. Since $\alpha$ is contact, by definition we know $\alpha \wedge d\alpha > 0$. In local coordinates this becomes

$$\begin{aligned}\alpha \wedge d\alpha =&(\beta + udz) \wedge d(\beta + udz)\\ =&\beta \wedge d\beta + \beta \wedge du \wedge dz + udz \wedge d\beta + udz \wedge du \wedge dz\\ =&(\beta \wedge du + ud\beta) \wedge dz > 0\end{aligned}$$

so that $\beta \wedge du + ud\beta > 0$ is an area form on $\Sigma$. For $p \in \Gamma_\xi := u^{-1}(0)$, then

$$\beta_p \wedge du_p + u(p)d\beta_p = \beta_p \wedge du_p + 0 > 0,$$

so we know that $du_p \neq 0$ on $\Gamma_\xi$. I.e., 0 is a regular value of $u$. $\square$

**Lemma 1.7.** $\Gamma_\xi$ is transverse to $\mathcal{F}_\xi$.

**Proof.** Suppose that $\Gamma_\xi$ and $\mathcal{F}_\xi$ are not transverse. Then there exists a point $p \in \Gamma \cap L$ for a leaf $L \subset \mathcal{F}$ and a vector $w_p \in T_p\Gamma \cap T_pL$. We have the following information:

- $v_p$ is transverse to $\Sigma$, so $w_p \neq v_p$, and $\alpha_p(v_p) = 0$ since $p \in \Gamma$.
- $\alpha_p(w_p) = 0$ since $w_p \in T_pL \subseteq \xi$.

So, if we can show that $(d\alpha_p)(w_p, v_p) = 0$, then this will violate the fact that $\alpha$ is a contact structure. We calculate that

$$\begin{aligned}\mathcal{L}_v\alpha = f\alpha \implies& \iota_v \circ d\alpha + d \circ \iota_v\alpha = f\alpha\\ \implies& \iota_{v_p} \circ d\alpha_p + d \circ \iota_{v_p}\alpha_p = f\alpha_p\\ \implies& (\iota_{v_p} \circ )\alpha_p + 0 = f\alpha_p\\ \implies& \iota_{v_p}(d\alpha_p) = f\alpha_p.\end{aligned}$$

But since $\alpha_p(w_p) = 0$,

$$\begin{aligned}&\implies \iota_{w_p}\iota_{w_p}(d\alpha_p) = f\alpha_p(w_p) = 0\\ &\implies (d\alpha_p)(w_p, v_p) = 0,\end{aligned}$$

as required. $\square$

**General Idea.** The characteristic foliation can be thought of as a family of "flow lines" on the surface, while the dividing set consists of curves where the flow lines abruptly change direction. Since this change must occur transversely, the dividing set is naturally embedded within the surface.

**Exercise.** Compute $\Gamma_\xi$ for $D^2 \times S^2$, $\Sigma = T^2$, using the radial vector field as a contact vector field with respect to which $\Sigma$ is convex.

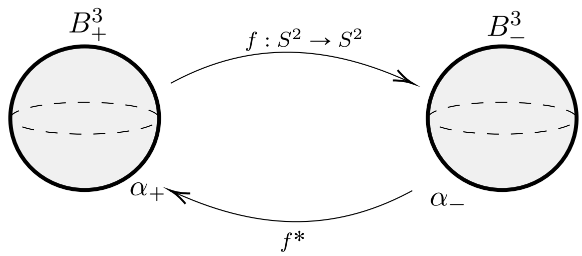
**Theorem 1.8.** *If $\mathcal{F}_1$, $\mathcal{F}_2$ are characteristic foliation for a surface $\Sigma$ with respect to contact structures $\xi_1$ and $\xi_2$, then there is a "nice" isotopy $\phi$ of $\Sigma \subset \nu(\Sigma)$ such that $\mathcal{F}_{\phi(\xi_1)} = \phi(\mathcal{F}_{\xi_2})$*

## 1.2 Worked Example of a Convex Surface

Reminder of last time:

**Question.** How do we glue contact manifolds?

Recall the first picture of gluing we described:




This was our toy case for gluing two contact manifolds along their boundary. In this case, our contact 3-manifolds are 3-balls each equipped with the standard contact structure (with opposing orientations) and we can glue these two contact manifolds together by finding some diffeomorphism of the boundary that "transports" one contact structure to another. That is, we find a diffeomorphism $f : S^2 \to S^2$ such that $f^*(\alpha_-)|_{S^2} = \alpha_+|_{S^2}$. The fact that $f$ is a diffeomorphism of the boundary allows us to conclude that the resulting manifold is smooth, and the fact that $f$ is a contactomorphism on the boundary allows us to smoothly identify the contact structures of each ball along the boundary. In short, given such an $f$, gluing is possible. However, this requires a lot of work to find when your gluing surface is not just $S^2$!

**Last time:** we talked about a few improvements and reductions we can make to this construction:

1. Characteristic foliations. By introducing the notion of a characteristic foliation of the surface $\Sigma$ we wish to glue along, we found that less information was needed to specify a gluing map than before. It turns out that if $\xi_1$ and $\xi_2$ are contact structures on some 3-manifold $M$, then for a surface $\Sigma \subset M$, if write out the characteristic foliations and find a diffeomorphism $f : \Sigma \to \Sigma$ such that $f_*(\mathcal{F}_{\xi_1}) = \mathcal{F}_{\xi_2}$, then we know that $f$ extends to a contactomorphism on a neighborhood of $\Sigma$. In other words, the characteristic foliation determines the contact structure in a small neighborhood of the surface $\Sigma$. How is this helpful? Well, what this means is that if you have two surfaces that admit the same characteristic foliation, then there is a contactomorphism on neighborhoods of those surfaces, which we can glue the two ambient manifolds together along in a manner that induces a contact structure.

2. Further, we also saw that we can do even better with dividing sets, given that $\Sigma$ is a convex surface with respect to a contact vector field. It turns out that, given two surfaces with the same dividing set, their characteristic foliations are the same up to contact isotopy.

**Today:** we will work through the specific example of the solid torus.

**Example.** Recall that the solid torus is given by $D^2 \times S^1 = \{(x, y, \theta) \in \mathbb{R}^2 \times S^1 \mid x^2 + y^2 = 1\}$. Then $D^2 \times S^1$ is a contact 3-manifold, with contact form $\lambda = \cos(\theta)dx - \sin(\theta)dy$

Let $V$ be the vector field on $D^2 \times S^1$ given by $V = x\partial_x + y\partial_y$. Recall that $V$ is $v$ is contact if $\mathcal{L}_v\lambda = f\lambda$ for $f \in C^\infty(M)$. Note that this is equivalent to $f_*(\xi) = \xi$.
Our goal is to compute the characteristic foliation and dividing set of $T^2 \subseteq D^2 \times S^1$. We will do this in the following steps:

1. Show that $T^2 = \partial(D^2 \times S^1)$ is convex, i.e.

   (a) Show $V$ is contact, and

   (b) Show that $V$ is transverse to $T^2$.

2. Draw the characteristic foliation $\mathcal{F}_\xi$ on $T^2$

3. Compute the dividing set $\Gamma_\xi$.

**Proof.**

1. (a) Recall that $V$ is contact if $\mathcal{L}_V\lambda = \iota_V \circ d\lambda + d \circ \iota_V \lambda = \lambda$. We calculate

$$\begin{aligned}\iota_V \circ d\lambda =& d(x\cos(\theta) - y\sin(\theta))\\ =& d(x\cos(\theta) - y\sin(\theta))\\ =& \cos(\theta)dx - x\sin(\theta)d\theta - \sin(\theta)dy - y\cos(\theta)d\theta\\ =& \lambda - (x\sin(\theta) + y\cos(\theta))d\theta\end{aligned}$$

and

$$\begin{aligned}d \circ \iota_V \lambda =& \iota_V(-\sin(\theta)d\theta \wedge dx - \cos(\theta)d\theta \wedge dy)\\ =& -\iota_V(\sin(\theta)d\theta \wedge dx + \cos(\theta)d\theta \wedge dy)\\ =& -(-x\sin(\theta)d\theta - y\cos(\theta)d\theta)\\ =& -(-x\sin(\theta)d\theta - y\cos(\theta)d\theta)\\ =& (x\sin(\theta) + y\cos(\theta))d\theta\end{aligned}$$

From which we can read off that $d \circ \iota_V\lambda$ cancels out the second term in the expansion of $\iota_v \circ d\lambda$, so $\mathcal{L}_V\lambda = \lambda$. That is, $V$ is contact for $\lambda$.

(b) To show that $V$ is transverse to $T^2$, we must first find a parametrization of the tangent space of $T^2$. Recall that

$$\begin{aligned}\Sigma := T^2 = \partial(D^2 \times S^1) &= \partial\{(x, y, \theta) \in \mathbb{R}^2 \times S^1 \mid x^2 + y^2 \leqslant 1\}\\ &= \{(x, y, \theta) \in \mathbb{R}^2 \times S^1 \mid x^2 + y^2 = 1\}.\end{aligned}$$

Change coordinates $\phi : \psi \mapsto (\cos(\psi), \sin(\psi)) = (x, y)$ so that

$$T^2 = \{(\psi, \theta) \in S^1 \times S^1\}.$$

The tangent space is then generated by $\phi_*(\partial_\psi), \partial_\theta$, i.e.

$$T_{(\psi,\theta)}\Sigma = \left\langle \begin{pmatrix}-\sin(\psi)\\ \cos(\psi)\\ 0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\rangle = \langle -y\partial_x + x\partial_y, \partial_\theta\rangle. \tag{*}$$

For which we can see that certainly $V = x\partial_x + y\partial_y$ is transverse to both $-y\partial_x + x\partial_y$ and $\partial_\theta$ (their inner product is 0 wrt. the standard inner product on $\mathbb{R}^3$).

2. Recall that the characteristic foliation of $T^2$ is given by intersecting its tangent space with the contact structure of the ambient contact manifold. Since we're intersecting two 2-plane distributions on a 3-manifold, dimension counting tells us that generically the outcome will be a 1-dimensional vector bundle over $T^2$, with some (generically countable or finite) collection of singular points.

We may integrate this (ignoring singular points for now) to give rise to the characteristic foliation.

Recall that our contact form is $\lambda = \cos(\theta)dx - \sin(\theta)dy$, and hence our contact structure is spanned by

$$\xi_{(x,y,\theta)} = \left\langle \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \tag{**}$$

For which we can read off that the characteristic foliation is given by the integral curves of the vector field $\partial_\theta$

3. From (*) and (**), the dividing set is precisely the points $p \in \Sigma$ where $T_p\Sigma = \xi_p$. This parametrization makes it clear where this may happen: namely where

$$x = \pm\cos(\theta), \quad \text{and} \quad y = \mp\sin(\theta).$$

Putting all of this together, we obtain the following picture:

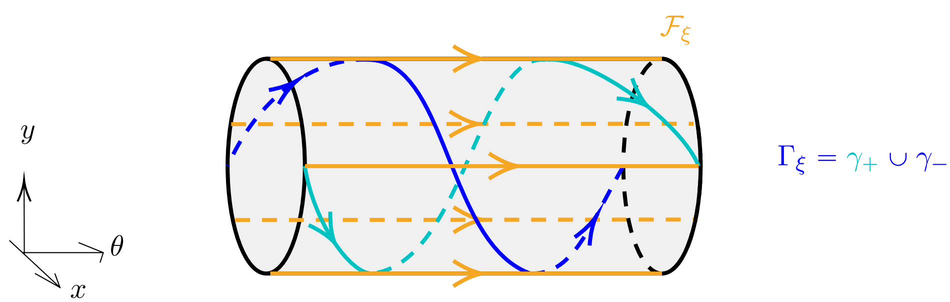



Figure 1: Characteristic foliation of $T^2$ in orange, with dividing curves in blue. Here, we identify the left end of the cylinder with $\theta = 0$, and the right end with $\theta = 2\pi$, so that each blue curve closes up under the quotient to give a $(1, 1)$ curve.

□

## 1.3 Dividing Sets Determine Contact Structure

So far, we've defined characteristic foliations, contact vector fields, convex surfaces, and dividing sets. We've also looked at a specific example, $T^2 \subset S^1 \times D^2$. The construction of these was motivated by reducing the problem of gluing two 3-manifolds along a common surface to something smaller - in this case matching up their dividing sets. We need to still check that this is really enough, i.e. that specifying a dividing set is enough to specify the contact structure on the surface and moreover on a tubular neighborhood of the surface. That way, if we can match up two dividing sets of surfaces in some fashion, then there will be an extension of this matching to

a tubular neighborhood of the surface. This will allow us to identify the contact structures over the gluing regions between the manifolds.
In order to talk about dividing sets giving rise to convexity of a surface, we need to first introduce a notion of a dividing set that is independent of them.

**Definition 1.9.** Let $\mathcal{F}$ be a singular 1-dimensional foliation on a closed surface $\Sigma \subset (M^3, \xi)$. Then a multicurve $\Gamma$ **divides** $\mathcal{F}$ if:

(a) $\Gamma$ is transverse to the foliation $\mathcal{F}$

(b) There exists an area form $\Omega$ and a vector field $X$ defining $\mathcal{F}$ such that

- $\mathcal{L}_X \Omega \neq 0$ on $\Sigma \backslash \Gamma$, and
- $\Sigma_\pm := \{\pm \mathrm{div}_\Omega(X) > 0\}^*$, $\Sigma \backslash \Gamma = \Sigma_+ \cup \Sigma_-$, and $X$ points out of $\Sigma_+$ along $\Gamma$.

Let's break down what this definition is saying: the dividing set $\Gamma$ of the characteristic foliation $S_\xi$ is transverse to $\mathcal{F}_\xi$, so this is covered in condition (a). The second part describes the complement of $\Gamma$ as two disjoint regions - ones where the characteristic foliation is flowing away and towards $\Gamma$ respectively.

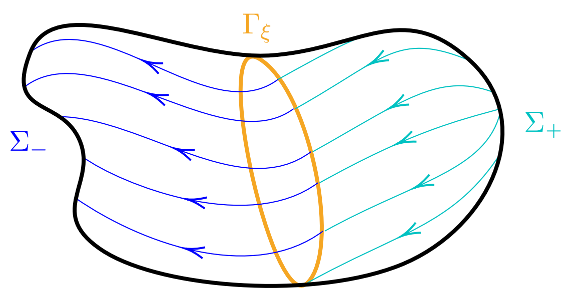



Figure 2: $\Sigma = \Sigma_- \cup_{\Gamma_\xi} \Sigma_+$. Note that on $\Sigma_+$, the characteristic foliation flows towards $\Gamma_\xi$ in positive time.

*Recall that the **divergence** $\mathrm{div}_\Omega(X)$ of a vector field $X$ on $\Sigma$ with respect to the area form $\Omega$ is defined by $\mathcal{L}_X(\Omega) = \mathrm{div}_\Omega(X)\Omega$ (remember here $\Omega$ is in the top cohomology class and so all area forms are related by some multiple of a function). Using the Cartan formula we can rewrite this as $\mathrm{div}_\Omega(X) = d(\iota_X \Omega)$.

There are two important theorems that nail down how dividing sets capture the information of the characteristic foliation of a convex surface:

**Theorem 1.10** ([Gei08], Theorem 4.8.5)**.** *Let $\Sigma$ be a surface in $(M^3, \xi)$.*

*(a) $\Sigma$ is convex if and only if $\mathcal{F}_\xi$ is divided by a collection of embedded circles $\Gamma$ on $\Sigma$.*

*(b) $\Gamma$ is determined by $\mathcal{F}_\xi$ up to isotopy through curves transverse to $\mathcal{F}_\xi$. In other words, given two contact vector fields $v_0, v_1$ for which $\Sigma$ is convex, one can find an isotopy $\psi_t : S \to S$ such that $\psi_1(\Gamma_0) = \Gamma_1$.*

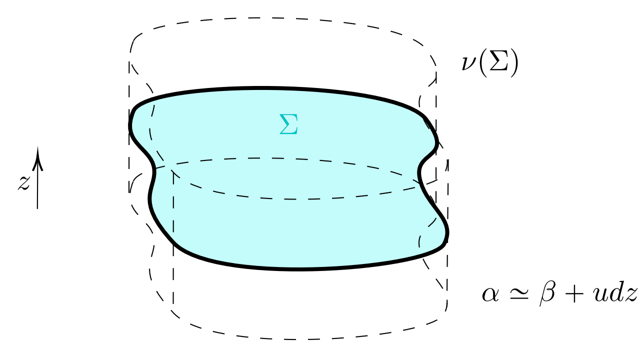
**Theorem 1.11** ([Gei08], Theorem 4.8.11)**.** *Let $\Sigma \hookrightarrow (M^3, \xi)$ be convex, and let $\mathcal{F}$ be any singular foliation divided by $\Gamma_\xi$. Then for any tubular neighborhood of $\Sigma$, $\nu\Sigma$, there exists an isotopy $\psi_t : \Sigma \to \nu\Sigma$ through convex embeddings such that $(\psi_1(\Sigma))_\xi = (\psi_1)_* \mathcal{F}$.*

**General Idea.** Theorem 1.10 says that convexity is determined by a dividing set - so if you can find a set that does what Definition 1.9 says, then you know that $\Sigma$ must admit a contact vector field. Moreover, this choice of contact vector field is unique up to isotopy. So the dividing set is not determined by your choice of contact vector field, but rather the underlying contact structure of the ambient manifold.

**Remark 3.** Let $\Sigma \hookrightarrow (M^3, \xi)$ be a surface, and let $\mathcal{F}_\xi$ denote its characteristic foliation. Then $\mathcal{F}_\xi$ determines $\xi$ in a neighborhood of $\Sigma$.

**Proof.** This is a special case of Giroux's theorem, Thm 2.5.22 in [Gei08]. Roughly, the theorem says that given a diffeomorphism between surfaces $\phi : \Sigma_0 \to \Sigma_1 \subset Y$ such that $\phi_*(\Sigma_{0,\xi_0}) = \Sigma_{1,\xi}$, then one may extend the $\phi$ to a contactomorphism of a neighborhood of each surface.

- Assume that a contact form in a neighborhood of each surface can be written as $\alpha_0 = \beta_0 + u_0 dz$ and similarly for $\alpha_1$, where $\beta_0$ is a smooth $z$-family of 1-forms and $u_0$ a smooth $z$-family of functions. Then, the characteristic foliation is given as $\ker(\beta_0)$ restricted to $\Sigma$.



- Extend $\phi$ to a neighborhood of each $\Sigma$, and consider the 1-forms $\alpha_0$ and $\phi^*\alpha_1$ restricted to the neighborhood of $\Sigma_0$.
- We can then run a Moser-type argument, observing that for sufficiently small neighborhoods, the convex-linear combination of $\alpha$ and $\phi^*(\alpha_0)$ are also contact on a neighborhood of $\Sigma_0$, and cut out a smooth family of foliations of $\Sigma_0$:

$$\alpha_t := t\phi^*\alpha_1 - (1-t)\alpha_0.$$

- More precisely, using the Moser trick, we find an isotopy $\psi : M \times [0,1] \to M$ such that $\psi_t^*\alpha_t = \lambda_t\alpha_0$ for some family of smooth functions $\lambda_t : M \to \mathbb{R}$. Then $\psi_1 \circ \phi : \nu\Sigma_0 \to \nu\Sigma_1$ is the desired contactomorphism.

Of course, this proof relies on the existence of such a diffeomorphism taking one characteristic foliation to another. In the set up of our remark, however, we do not have this piece of information. Since our characteristic foliations may be singular, it is not that easy to cook up a diffeomorphism that respects the foliation,though they may be homeomorphic. Sections 4.6-4.8 of [Gei08] discusses how one can eliminate certain singularities from occuring, outside of which one does not run into problems creating the diffeomorphism. □

We will now prove Theorems 1.10 and 1.11.

**Proof.** *Of Theorem 1.10.*

(a) The forwards direction follows by definition. If $\Sigma$ is convex, then we may define the characteristic foliation of $\Sigma$ and the dividing set as in Definitions 1.2 and 1.5. Then $\Gamma_\xi$ is a collection of embedded circles on $\Sigma$, and $\mathcal{F}_\xi$ is divided by $\Gamma_\xi$ by Lemma 1.7. To show the other direction, let us suppose that $\mathcal{F}_\xi$ is divided by a collection of embedded circles on $\Sigma$, as in Definition 1.9. Let $\Omega$ and $X$ be as in the definition. We need to show that $\Sigma$ is convex, and so must cook up some contact vector field $v$ that is transverse to $\Sigma$.

**Idea:** To do so, we'll consider a tubular neighborhood $\Sigma \times \mathbb{R}_z$, and construct a new contact form $\tilde{\alpha}$ such that $\tilde{\alpha}$ and $\alpha$ induce the same characteristic foliation on $\Sigma$. In particular, we will construct $\tilde{\alpha}$ to be $\mathbb{R}$-invariant, so that $v = \partial_z$ is a contact vector field that is naturally transverse to $\Sigma$. This will ensure that $\Sigma$ is convex with respect to $\ker \tilde{\alpha}$. And since $\tilde{\alpha}$ and $\alpha$ share the same characteristic foliation, then by Remark 3, $\Sigma$ will also be convex with respect to $\ker \tilde{\alpha} = \xi$.

Let $\beta = \iota_X \Omega$ and set $\tilde{\alpha} = \beta + u dz$. Then $\tilde{\alpha}$ is an $\mathbb{R}$-invariant 1-form on $\Sigma \times \mathbb{R}$, where $u : S \to \mathbb{R}$ is some smooth function that does not depend on $z$. We identify $i : \Sigma \simeq \Sigma \times \{0\} \hookrightarrow \Sigma \times \mathbb{R}$. Note that $\ker i^*(\tilde{\alpha}) = \ker \beta|_{\Sigma \times \{0\}} = \mathcal{F}_\xi$. In other words, $\alpha$ and $\tilde{\alpha}$ induce the same characteristic foliation on $\Sigma$. We have yet to define $u$, however, and the last piece of the puzzle we need is that $\tilde{\alpha}$ is contact. Then, we can appeal to Remark 3, and apply the logic of above.

Let us rewrite the contact condition in terms of $u$ and $X$ to see how this may inform our choice:

$$\begin{aligned}\tilde{\alpha} \wedge d\tilde{\alpha} =& (\beta + udz) \wedge d(\beta + udz) \\ =& \beta \wedge d\beta + \beta \wedge du \wedge dz + udz \wedge d\beta + udz \wedge du \wedge dz \\ =& (\beta \wedge du + ud\beta) \wedge dz > 0\end{aligned}$$

so that $\tilde{\alpha}$ is contact iff $(\beta \wedge du - ud\beta)$ is a positive area form on $\Sigma$. In the last line, note that $d\beta$ is a 2-form, and $dz$ is a 1-form, so that $dz \wedge d\beta = d\beta \wedge dz$. Now, $\beta := \iota_X \Omega$, so we may rewrite this condition as

$$\begin{aligned} & \beta \wedge du + ud\beta > 0 \\ \Leftrightarrow \quad & (\iota_X \Omega) \wedge du + ud(\iota_x \Omega) > 0 \\ \Leftrightarrow \quad & -X(u)\Omega + u\mathrm{div}_\Omega(X)\Omega > 0 \\ \Leftrightarrow \quad & u\mathrm{div}_\Omega(X) - X(u) > 0.\end{aligned}$$

We set $u \equiv \pm 1$ on $\Sigma_\pm$. Then the contact condition holds on these pieces. We now just have to interpolate over $\Gamma$. [Gei08] goes into detail about how one can do this on page 234 (hopefully I'll have time to add these details later).

(b) Suppose that $\Sigma$ is a convex surface with respect to two different contact vector field $v_0$ and $v_1$. To prove the theorem, we'd like to show that there is an isotopy $\psi_t : \Sigma \to \Sigma$ taking one dividing set $\Gamma_0$ to another $\Gamma_1$. Moreover, we have to show that for each $t \in [0,1]$, $\psi_t(\Gamma_0)$ is transverse to $\mathcal{F}_\xi$. Assume that the vector fields $v_0$ and $v_1$ are compactly supported near $\Sigma$, so that we may use their respective flows to identify a neighbourhood of $\Sigma$ (in two different ways) with $\Sigma \times \mathbb{R}$. Under these identifications, the contact structure $\xi$ gives us two vertically invariant contact structures $\xi_0$ and $\xi_1$ on $\Sigma \times \mathbb{R}$, both inducing the characteristic foliation $\mathcal{F}_\xi$ on $\Sigma \equiv \Sigma \times \{0\}$. Thus, wlog we may write $\xi_i = \ker(\beta + u_i dz)$. Then $\Gamma_i$ is the zero set of the function $u_i$. Moreover, the contact condition for fixed $\beta$, is convex in $u$, so we have the linear interpolation of vertically invariant contact forms $\beta + ((1-t)u_0 + tu_1)dz$, $t \in [0,1]$, on $\Sigma \times \mathbb{R}$. This gives us an $\mathbb{R}$–equivariant isotopy $\psi_t = \phi_t \times \mathrm{id}_\mathbb{R}$. Hence, the isotopy $\phi_t : \Sigma \to \Sigma$ moves the zero set of $u_0$ to the zero set $\Gamma_1$ of $u_1$, via dividing sets of $\mathcal{F}_\xi$ given by the zero sets of $(1-t)u_0 + tu_1$, all of which constitute collections of curves transverse to $\mathcal{F}_\xi$.

$\square$

**Proof.** *Of Theorem 1.11.* Let $\Sigma \hookrightarrow (M^3, \xi)$ be a convex surface, and let $\mathcal{F}$ be any singular foliation divided by $\Gamma_\xi$. We can identify a neighborhood of $\Sigma$ contactomorphically with $\Sigma \times \mathbb{R}$, equipped with a vertically invariant contact structure $\xi \simeq \xi_0 = \ker \alpha_0$, with $\alpha_0 = \beta_0 + u_0 dz$. Choose a collection $A$ of closed annuli around the dividing curves in such a way that the flow lines of both $\mathcal{F}_\xi$ and $\mathcal{F}$ foliate each annulus by line segments from one boundary component to the other (and transverse to the boundary). We assume that we are in the set up of Definition 1.9, so that we have an area form $\Omega$ on $\Sigma$ and a vector field $X_0$ directing $\mathcal{F}_\xi$, so that $\alpha_0 = \iota_{X_0}\Omega + u_0 dz$ with $u_0 \equiv \pm 1$ on $\Sigma_\pm \backslash A^o$, and $\pm \mathrm{div}_\Omega(X) > 0$ on $\Sigma_\pm$. We also assume that $\Gamma_\xi = u_0^{-1}(0)$.
By assumption, $\mathcal{F}$ is divided by $\Gamma_\xi$. Hence, there exists a vector field $X'$ and an area form $\Omega'$ on $\Sigma$ such that $\pm \mathrm{div}_{\Omega'}(X') > 0$ on $\Sigma_\pm$. Moreover, since we're on a closed surface, $\Omega' = g\Omega$ for some nonvanishing function $g : \Sigma \to \mathbb{R}$, $g > 0$.
Now,

$$g \mathrm{div}_{g\Omega}(X') = \mathrm{div}_\Omega(gX'),$$

so that if we set $X_1 := gX'$, then $\mathrm{div}_\Omega(X_1) > 0$ whenever $\mathrm{div}_{g\Omega}(X') > 0$. Hence, $X_1$ is a vector field such that $\pm \mathrm{div}_\Omega(X_1) > 0$ on $\Sigma_\pm$. In particular, $X_1$ also directs $\mathcal{F}$.
Following the standard Moser trick, our goal is to find some 1-parameter family of one forms $\alpha_t$ such that $\alpha_0$ is our original contact form, and $\ker \alpha_1 =: \xi_1$ has characteristic foliation of $\Sigma$ precisely $\mathcal{F}$. We define $X_t := (1-t)X_0 + tX_1$, and set $\alpha_t := \iota_{X_t}\Omega + u_t dz$, for some $u_t$ satisfying the same conditions as $u_0$. In particular, note that $X_1$ defines $\mathcal{F}$, so that actually $\mathcal{F} = \mathcal{F}_{\xi_1}$.
We integrate the family $X_t$ to give an isotopy $\psi_t : M \to M$ such that $(\psi_t)^* \xi_t = \xi_0$. Moreover, $\psi_t$ restricts to a neighborhood of $\Sigma$, $\psi_t : \Sigma \to \nu(\Sigma)$ and we have that

$$\begin{aligned}
(\psi_1(\Sigma))_{\xi_0} =& (T\psi_1(\Sigma) \cap \xi_0) \\
=& (\psi_1)_*(T\Sigma) \cap \xi_0 \\
=& (\psi_1)_*(T\Sigma) \cap (\psi_1)_*(\xi_1) \\
=& (\psi_1)_*(T\Sigma \cap \xi_1) \\
=& (\psi_1)_*(\mathcal{F}_{\xi_1}) \\
=& (\psi_1)_*(\mathcal{F}),
\end{aligned}$$

as required. □

## 1.4 Convex Surfaces in Action

### Example 1: Qiuyu

$(M_1, \xi_1)$ and $(M_2, \xi_2)$, we can take the connect sum $(M_1 \# M_2, \xi_1 \# \xi_2)$. Take a local ball, and you can arrange so that boundary is convex and the dividing set is just a meridian.

### Example 2: Jianru

Let $M$ be a connected 3-mfld that is compact, and let $\Gamma \subset \partial M$. Let $\mathcal{F}$ be a singular foliation divided by $\Gamma$. Define

$$\mathrm{Cont}(M, \mathcal{F}) = \{\text{all contact structures } \xi \text{ on } M \mid \partial M \text{ is convex }, \xi_{\partial M} = \mathcal{F}\}$$

up to isotopy of $\xi$ relative to boundary.
Observe if $\mathcal{F}'$ is also divided by $\Gamma$, then Giroux flexibility says that the set $\mathrm{Cont}(M, \mathcal{F}')$ is in 1-1 correspondence. I.e. there's a bijection between these two sets.

### Example 3: Audrey

Crazy theorem of Gabai: if $\Sigma \subset M$ minimizes gens in its homology class, then there exists a taut foliation $\mathcal{F}$ such that $\Sigma$ is a leaf.
Proof: Sutured manifold hierarchies.
Convex manifold hierarchies were then established by Honda. Used to create hypertight Reeb flows Sutured ECH, HF, etc. Building Reeb flows
Colin and Honda - survey paper called "Foliations, contact structures and their interactions in dimension 3"

### Example 4: Elliot

How dividing sets can change - bypass

### Example 5: Robert

$\Sigma \neq S^2$ that is convex with dividing set $\Gamma$, then $\nu(\Sigma)$ is tight if and only if $\Gamma$ has no homotopically trivial curves. Giroux's criterion for tightness. Can use this theorem with one of Eliashberg Ko Honda paper used to classify tight contact structures on Torus. classify all tight contact structures on using Eliashberg's theorem.

### Example 6: Nancy

Non simple Legendrian knots

# 2 Open Book Decompositions

## 2.1 First Definitions of an Open Book

**Definition 2.1** (Open Book Decomposition)**.** Open book decomposition of a 3-manifold $M$ is a representation of $M$ comprising a link $L \subset M$ and a fibration $\pi : M\backslash L \to S^1$ such that any fiber of the fibration is a Seifert surface $S$ for the link, i.e. $\partial S = L$.

**Example.** The following fibration of $S^3$ over the Hopf link that is not an open book decomposition, since the boundary is not the whole link $L$.

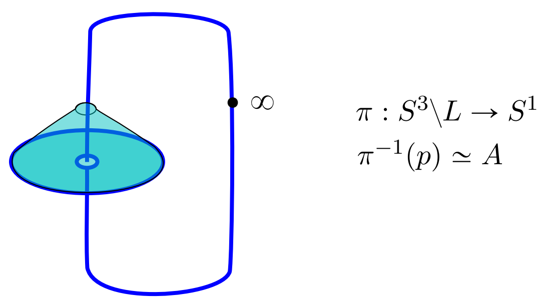
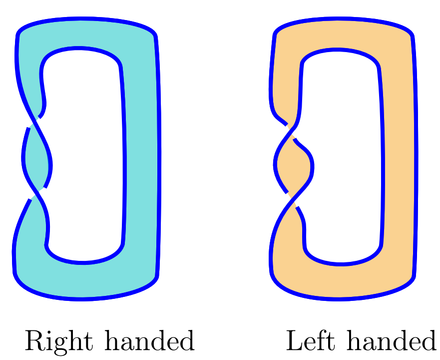



Figure 3: Hopf Link (blue) in $S^3$.

**Example.** We can however construct two other fibrations of $S^3$ over the Hopf link which *are* open book decompositions. These are called the **right handed OBD** and the **left handed OBD**.




This picture can also be described algebraically. Consider $S^3$ in $\mathbb{C}^2$ as

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\} = \{(r_1, r_2, \theta_1, \theta_2) \mid r_1^2 + r_2^2 = 1\}$$

We can identify the Hopf link with the subspace $H = \{r_1 r_2 = 0\}$. From here, we can explicitly write down a fibration $\pi : S^3 \backslash L \to S^1$ which sends

$$\pi(r_1, \theta_1, r_2, \theta - 2) = \theta_1 - \theta_2.$$

One can check that all singular points lie on $L$, which is not in the domain. Another fibration is given by the projection map

$$\pi(r_1, \theta_1, r_2, \theta - 2) = \theta_1 + \theta_2.$$

These projection maps correspond to the left and right handed fibrations as above. T Add Elliot's picture about the hope fibration of $S^3$ over $S^2$ and you can see the preimage of each line is an annulus, and you get right/left handed depended on which hopf fibration you choose.
Another picture: express $S^3 = S^1 * S^1$, where $*$ denotes join.

$X \star Y \ := \ X \sqcup Y \cup (X \times Y \times I)$ $\qquad\qquad$ $S^3 = S^1 \star S^1$

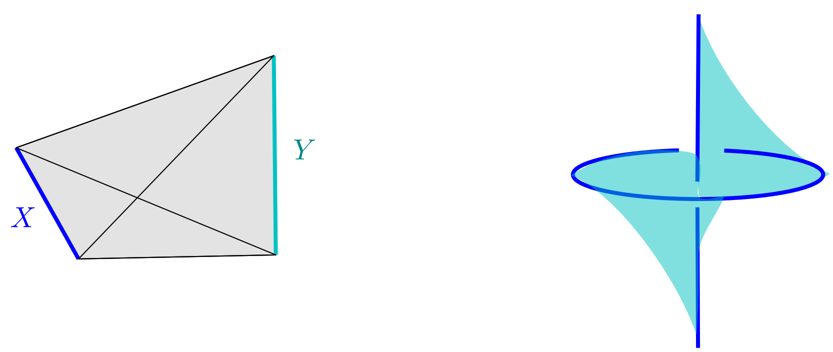



Figure 4: Right and left handed open book decomposition of $S^3 \simeq S^1 \star S^1$.

We recover the right handed fibration by letting $\theta_1 \in [0, 2\pi]$ and $\theta_2 = \theta_1 + \tau$, with $\tau \in [0, 2\pi]$. You recover the left-handed fibration by letting $\theta_2 = \tau - \theta_1$.

**Remark 4.** For crossings $\leqslant 13$, a knot if fibered if and only if the Alexander polynomial is monic. In this case, the genus of the fiber is g, where the Alexander polynomial has degree 2g.

**Theorem 2.2.** *Any closed, orientable connected 3-manifold $M$ admits an open book decomposition. I.e. you can find a link $L \subset M$ along with a. fibration whose leaves have boundary $L$.*

**Definition 2.3** (Abstract open book)**.** Given a compact, connected, oriented surface with boundary $\Sigma$ and a diffeomorphism $\phi : \Sigma \to \Sigma$ which fixes pointwise $\partial\Sigma$, we can form a 3-manifold out of this data by constructing the mapping torus of $\phi$,

$$(\Sigma \times I)/((x, 1) \sim (\phi(x), 0))$$

which has torus boundary $= \partial\Sigma \times (I/\partial I)$. The surface $\Sigma$ is called the **page** of the open book, and the map $\phi$ is called the **monodromy** of the open book (or more specifically of the mapping torus). We would like to eliminate the boundary components, and so we Dehn-fill the mapping torus to kill $\{x\} \times S^1$ for $x \in \partial\Sigma$. After these two procedures to get a closed $M^3$. The core of the Dehn filling is called the **binding**. We denote an abstract open book by $(\Sigma, \phi)$, where $B$ is fixed by such a choice.

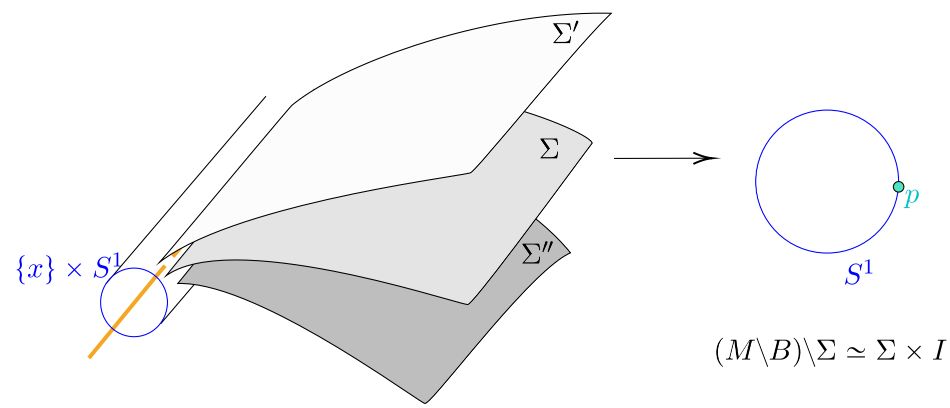



Figure 5: Schematic of OBD via mapping torus.

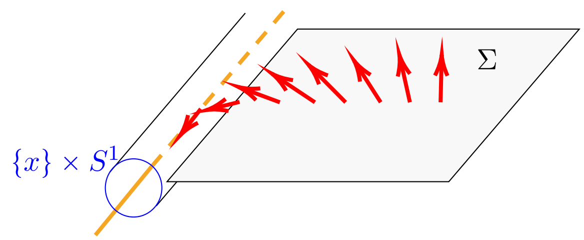
It is not hard to see that our first definition of an open book agrees with our new definition.
John Etnyre's handwritten notes contain good pictures of some examples including **Milnor fibrations** (need to add citation)

## 2.2 Supported contact structures

**Definition 2.4.** Let $(M, \xi)$ be a contact 3-manifold. The contact structure $\xi$ is said to be **supported by an open book of** $M$ (up to isotopy of $\xi$) if there exists a contact 1-form $\alpha$ for $\xi$ such that

- $d\alpha$ restricts to a positive volume form on each page of the decomposition $\Sigma$, with respect to a chosen orientation of the page.
- $\alpha$ is positive on the binding, with respect to the induced orientation of the page.

**Proposition 2.5.** *A contact structure $\xi$ on $M$ is supported by open book if and only if there exists (up to isotopy) a contact 1-form $\alpha$ such that the corresponding Reeb vector field $R_\alpha$ is tangent to the binding and transverse to every page.*




We can think of things in terms of the Reeb fields $R$. The condition that $d\alpha > 0$ is that the Reeb field $R$ is positively transverse to $P_c$. The Reeb field twists as you move along $P_c$ towards the binding. Around the binding, the Reeb field circles around, with ever decreasing slope the closer you move to the binding. With this construction, the Reeb field is parallel to the binding.
If we choose two pages $\Sigma_c$ and $\Sigma_{-c}$ and union them together, we see that the resulting surface $\Gamma_c = \Sigma_c \sqcup \Sigma_{-c}$ is **convex**, with dividing set exactly the binding of the open book.

**Theorem 2.6** (Thurston- Winkelnkember)**.** *Any open book supports a contact structure.*

**Proof.** First, fix local coordinates of the neighborhood of the binding. Think of this as the torus used to Dehn fill the mapping torus of $\Sigma$, from the abstract open book decomposition construction $(\Sigma, h)$. Call the binding direction $\theta$, the radial direction $r$, and the angular direction around the binding $t$.
Find a 1-form $\alpha$ on $\Sigma$ such that

1. $d\alpha > 0$ on $\Sigma$
2. $\alpha = (2 - r)d\theta$ near $\partial\Sigma$.

The set of all $\alpha$ such that the above holds is convex in $\Omega^1(\Sigma)$. Furthermore, $h^*\alpha$ satisfies conditions 1,2. Therefore, $t\alpha + (1-t)h^*\alpha$ satisfies conditions 1,2. Choose $\beta \in \Omega^1(\Sigma \times I)$, defined by $\beta = t\alpha + (1-t)h^*\alpha$. Then $\beta$ induces a one-form on the mapping torus $T_h$. (We constructed $\beta$ to satisfy this). Furthermore, $\beta$ satisfies conditions 1,2. Construct the form $\beta + Kdt$ for $K >> 0$. Then, this is contact. Indeed, we compute $(\beta + Kdt) \wedge d\beta = \beta \wedge d\beta + Kdt \wedge d\beta$. The first term may very well be negative or zero. The second term is always positive, because we fix $d\beta > 0$. Therefore, for $K$ large enough, the associated volume form must be positive. Now we extend $\beta$ to a neighborhood of the binding. To do this, we find a contact form on the solid torus such that it agrees at $r = 1$ with the local model we imposed for

$\beta$ on the mapping torus. Looking back to condition 1, we know $\beta = (2-r)d\theta + Kdt$ near $r = 1$. For our local model, we take

$$\beta = f(r)d\theta + g(r)dt \quad r \in [0,1]$$

We need these functions $f, g$ to satisfy a number of conditions. For this to be a contact form locally, we check that

$$\beta \wedge d\beta = (fg' - f'g)d\theta \wedge dr \wedge dt.$$

We need

- $fg' - f'g > 0$ so that $\beta$ is contact.
- $f(r) = 2 - r$, and $g(r) = K$ near $r = 1$ so that local models match up.
- $f(r) > 0$ near $r = 0$ so that $\alpha(L) > 0$, i.e. that the binding is contact.
- $f'(r) > 0$ everywhere so that $d\beta > 0$ on leaves.
- $g(r) \sim r^2$ near $r = 0$ so that $g(r)dt$ vanishes as $r \to 0$. But remember that $rdt$ is the nonsingular part, so for this to vanish we need to use $r^2dt$.

To satisfy all of these, we pick $f(r) = 2 - r^2$ and $g(r) = r^2$. We can check that this satisfies all the conditions above.
Observe that the constructed local form around the binding has the binding as a Reeb orbit.

□

**Theorem 2.7** (Giroux). *Any two contact structures $\xi_1, \xi_2$ supported by the same open book are isotopic via a path of contact structures.*

**Proof.** Let $\beta_0, \beta_1$ be any two contact 1-forms carried by the open book $(L, \pi)$. The convex combination of $\beta_0$ with $\beta_1$ does not work. But, we can force things to be contact by adding in a term $Kdt$ for $K >> 0$. choose a function $f(r)$ such that $f(r) \sim r^2$ near $r = 0$, and $f(r) = 1$ when $r \geqslant 1$. Then, take the contact form

$$\beta^K = \beta + f(r)Kdt$$

Computing the contact volume, we see

$$\begin{aligned}\beta^K \wedge d\beta^K &= (\beta + fKdt) \wedge (d\beta + f'Kr \wedge t) \\ &= \beta \wedge \beta + Kft \wedge \beta + f'K\beta \wedge r \wedge t\end{aligned}$$

The first term is always positive because $\beta$ is contact. The second is strictly positive outside of the binding (as $\beta$ is positive on the pages), and nonnegative on $M$. The third term is strictly positive near the binding, and nonnegative on $M$. All together, $\beta^K$ is contact. Then, the interpolation

$$t\beta_0^K + (1-t)\beta_1^K$$

is contact for $K >> 0$. Therefore, we can interpolate between contact forms supported by an open book. □

# 3 Giroux Correspondence

The last thing we'll do in this seminar is discuss Giroux correspondence, a profound result that completely pins down the relationship between open book decompositions and supported contact structures of a manifold. The statement is the following.

**Theorem 3.1** (Giroux Correspondence). *Let $M^3$ be a closed, oriented 3-manifold. Then there is a correspondence*

$$\{\text{Oriented } \xi \text{ on } M\} \big/ \{\text{isotopy}\} \longleftrightarrow \{\text{OBDs on } M\} \big/ \{+\text{ve stabilization}\}$$

Let's unpack what this statement is saying. We've already seen that any open book supports a contact structure, and that two contact structures supported by the same open book are isotopic via a path of contact structures. So, we've established that

$$\Phi : \{\text{OBDs on } M\} \big/ \{+\text{ve stabilization}\} \longrightarrow \{\text{Oriented } \xi \text{ on } M\} \big/ \{\text{isotopy}\}$$

is a function between sets. We'd like to show that this function is surjective an injective. In other words, the work that remains is to show that every contact structure is supported by an open book decomposition, and that these open book decompositions are unique up to stabilization.

**Theorem 3.2** (Giroux, 2000). *Let $(M, \xi)$ be a contact 3-manifold. Then $M$ admits an open book decomposition $(\Sigma, \phi)$ that supports $\xi$.*

**Proof.** I could just write the really shitty version where we don't explain how to isotope the CW complex, and then quote some big theorems. □

The result of theorem 3.2 tells us that the function $\Phi$ is surjective. We now need to prove that it is injective. The following discussion will give us some progress towards this.

## 3.1 Stabilization

The goal is to give some understanding of the following theorem.

**Theorem 3.3.** *The open books supporting a given contact 3-manifold $(M, \xi)$ are unique up to positive stabilization.*

Establishing injectivity of the map $\Phi$ is difficult. We will only show that two open book decompositions support the same contact structure if they are related by stabilizations. We will not discuss the "only if" version of this statement. Before we define stabilizations, here is an immediate observation of this fact.

**Example.** The standard contact structure on $S^3$ is supported by the positive Hopf link with its associated knot fibration, but not the negative Hopf link. So the positive and negative Hopf link fibrations are never positive stable-equivalent.

**Definition 3.4.** The **positive/negative stabilization** of an open book $(\Sigma, \phi)$ is the open book $S_{\pm}(\Sigma, \phi)$ where the pages are $\Xi$ with a one handle attached, and the monodromy is $\phi \circ \tau_{\pm}$, where $\tau_{\pm}$ is the left/right handed Dehn twist arond the core $C$ of the attached one-handle.

This is a special example of the **Murasugi sum**. For two abstract open book decompositions $(\Sigma_1, \phi_1))$ and $(\Sigma_2, \phi_2)$, we can add them together via the following process. Choose properly embedded arc $c_i$ on each $\Sigma_i$, and denote by and $R_i$ a rectangular neighborhood of $c_i$ in $\Sigma_i$. We form the Murasugi sum $(\Sigma_1 \phi_1) \star (\Sigma_2, \phi_2)$ as the open book decomposition with page $\Sigma_1 \cup_{R_1 \sim R_2} \Sigma_2$, and monodromy $\phi_2 \circ_1$, where we extend the diffeomorphism across the surface by the identity. The $\pm$ Stabilization is the Murasugi sum with the annulus as the page and with a right/left handed Dehn twist as the monodromy.
The resulting 3-manifold of the Murasugi sum is independent of $c_i$, but the open book decomposition *does* depend on the choice $c_i$. This is already clear in the case of $\pm$ stabilizations. Observe that stabilization via different curves can change the boundary components, and so can change the topology of the surface $S_{\pm}\Sigma$. yet we will see, the 3-manifolds are the same.

**Theorem 3.5** (Gabai, 1983)**.** *If $(M_i, \xi_i)$ are supported by the open book $(\Sigma_i, \phi_i)$ for $i = 1, 2$, then $Y_1 \# Y_2$ with its connect sum contact structure $\xi_1 \# \xi_2$ is supported by the open book $(\Sigma_1, \phi_1) \star (\Sigma_2, \phi_2)$.*

**Remark 5.** To connect sum contact structures relative to chosen base points, we first must choose a Darboux chart around each point. A sufficiently small 3-ball will be tight, and so its boundary is a convex structure with a single dividing curve. By contact surgery, we can canonically glue the two manifolds together along these small 2-spheres. This defines the connect sum contact structure, up to isotopy.

**Corollary 3.6.** The stabilization corresponds to the connect sum witha 3-sphere with the Hopf open book decomposition, so it does not change the topological type of the manifold nor the contact structure:

$$M_{S_{\pm}(\Sigma,\phi)} \cong M_{(\Sigma,\phi)} \# M_{(S^3,\xi_0)} = M_{(\Sigma,\phi)}.$$

We sketch the proof of theorem 3.5.
**Proof.** We'll employ the following strategy. First, we'll draw $M_{(\Sigma_1,\phi_1)\star(\Sigma_2,\phi_2)}$ abstractly as a mapping torus that has been Dehn filled. Our goal will then be to find a 2-sphere in this three manifold along which we can cut so that each end is isomorphic to $M_{(\Sigma_i,\phi_i)}$ for each $i = 1, 2$. I.e., so that $M$ decomposes as a connect sum of these two abstract open book decompositions. To build such a 2-sphere, We'll first restrict ourselves to a neighborhood of a page, $\Sigma_1 \star \Sigma_2 \times I$. The rough idea is as follows. Here, we can find a surface $S$ with boundary given by 4 parallel strands, each cut out by some $p \times I$. After gluing in a neighborhood of the binding, we can find a disc with boundary given by any one of these strands, and by gluing them to $S$ we get a

sphere. (need to add figure) How can we be so sure that such a surface exists? Well, to build this surface, we can take our four parallel strands be the “corners” of the Murasugi sum - the places where the two surfaces’ boundaries are glued together. These four points can be paired up in two different ways - using the arc $c_1$ from $\Sigma_1$ or using the arc $c_2$ from $\Sigma_2$. On the lower half of the interval, fill in by pairing the points via $c_1$. After gluing in 4 discs along each of the boundary components, we get a 2-sphere. □

# Index



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